ON THE MOTION OF A SPHERE IN A PERFECT FLUID, WITH APPLICATION TO LIQUID HELIUM*

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Abstract.—Momentum relationships involved in the motion of a sphere through a perfect fluid are considered. The mechanism, by which the momentum given to a sphere by an external impulsive force is transferred to the container as a whole, is traced through in some detail, and energy and momentum relationships are discussed. Comparison is made with a ^{3}He atom or a roton moving in superfluid ^{4}He.

Superfluid helium behaves much like an incompressible inviscid fluid, and the motion of foreign molecules, electrons, and excitations through it has led to renewed interest in a problem of classical hydrodynamics, namely, the movement of a solid sphere in a perfect fluid. This problem has been worked out for a sphere in an infinite fluid and for a sphere at the center of a spherical box.[1] However, several interesting questions were not considered.

One especially interesting experiment is the production of excitations in superfluid helium by neutrons.[2] These are produced with definite energy and definite momentum, there being a relation between the energy ϵ and the momentum p, as shown roughly in Figure 1. If we imagine a container of liquid helium suspended by a string and being bombarded by a beam of neutrons at right angles to the string, it is clear that the momentum of the excitation produced will be ultimately transmitted to the container. A somewhat similar process would be the transfer of momentum to a sphere in a perfect fluid by a neutron which did not act on the fluid. An analysis of the mode of transfer of momentum to the container might, hopefully, offer some further physical insight into the process of excitation, especially of rotons, in liquid helium, even though this has been the subject of a detailed quantum mechanical investigation.[3] So let us consider the motion of a sphere of radius a located at the center of a large but finite spherical container of radius b filled with perfect fluid. Other geometrical arrangements could be considered, but they would be much more complicated, and the one suggested is sufficient to throw considerable light on the dynamics of the situation.

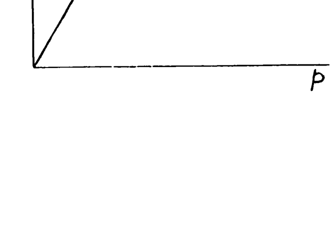

FIG. 1.—Relation between energy and momentum for excitations in liquid helium. The excitations with low p are sound waves (phonons), but the excitations near the minimum (rotons) have a localized character, and their motion through the superfluid, particularly, resembles to some extent that of a body moving through a perfect fluid.

We designate positions in the container by polar coordinates, r, θ, ϕ, with origin at the center of the container, and with the polar axis in the direction of the motion of the small sphere. The velocity $\mathbf{v}$ of the fluid at any point may be described in terms of a velocity potential Φ which will obey Laplace's equation. By symmetry Φ will not depend on the equatorial angle ϕ, so the Laplace equation takes the form

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right) + \frac{1}{\sin \theta}\frac{\partial}{\partial \theta}\left(\sin \theta \frac{\partial \Phi}{\partial \theta}\right) = 0 \tag{1}$$

with, of course,

$$\mathbf{v} = - \text{grad } \Phi. \tag{2}$$

At the surface of the moving sphere the normal component of $\mathbf{v}$ is equal to the normal component of the velocity $\mathbf{v}_0$ of the sphere. At the surface of the container the normal component of $\mathbf{v}$ vanishes. Thus the boundary conditions are

$$\text{at } r = b, \quad \partial\Phi/\partial r = 0, \tag{3a}$$

$$\text{at } r = a, \quad - \partial\Phi/\partial r = v_0 \cos \theta, \tag{3b}$$

where v_0 is the magnitude of $\mathbf{v}_0$.

A solution of equation (1) which conforms to the boundary conditions is[1]

$$\Phi = (Ar + B/r^2) \cos \theta, \tag{4}$$

with

$$A = v_0 a^3/(b^3 - a^3), \tag{5}$$

and

$$B = {}^1/_2 v_0 a^3 b^3/(b^3 - a^3). \tag{6}$$

With these values of A and B, equations (2) and (4) may be used to obtain the velocity $\mathbf{v}$. It is convenient to use the Cartesian components of $\mathbf{v}$, so we write

$$\Phi = Az + Bz/r^3. \tag{7}$$

Recalling that $\partial r/\partial x = x/r$, etc., we find

$$v_x = 3Bzx/r^5 = (3B/r^3) \cos \theta \sin \theta \cos \phi, \tag{8}$$

$$v_y = 3Bzy/r^5 = (3B/r^3) \cos \theta \sin \theta \sin \phi, \tag{9}$$

$$\begin{aligned} v_z &= 3Bz^2/r^5 - A - B/r^3, \\ &= (3B/r^3) \cos^2 \theta - A - B/r^3. \end{aligned} \tag{10}$$

The kinetic energy of the liquid plus that of the sphere of radius a having the same density ρ as the liquid is given by

$$E_k = {}^1/_2\, \rho({}^4/_3\, \pi\, a^3)v_0^2 + {}^1/_2 \rho \int_0^{2\pi}\int_0^{\pi}\int_a^b (v_x^2 + v_y^2 + v_z^2) r^2 \sin \theta\, dr\, d\theta\, d\phi. \tag{11}$$

Only v_z depends on A. Contributions from the terms $(2AB/r^3)$ $(3 \cos^2 \theta - 1)$

occurring in v_z^2 cancel since the average of $\cos^2\theta$ over the sphere is $^1/_3$. If $b \gg a$, then A^2 is negligible compared to B^2/r^6 except where $r \approx b$, but if b/a is large, these regions contribute negligibly, anyhow. Nevertheless, the contribution from A^2 can be included, and we obtain the known result

$$E_k = {^3/_2} \cdot {^1/_2}\, \rho({^4/_3}\, \pi\, a^3)v_0^2 \cdot b^3/(b^3 - a^3), \tag{12}$$

which, when $b \rightarrow \infty$, approaches the kinetic energy of a body with 3/2 the mass of the sphere moving with velocity $\mathbf{v}_0$.

The momentum of the fluid has apparently not been discussed in the classical literature. The total momentum, p, of the fluid should, if the container is stationary, be equal and opposite to that of the moving sphere, if the latter has the same density as that of the fluid, for in this case there should be no net momentum. We need consider only v_z and

$$p = \rho \int_0^{2\pi}\int_0^{\pi}\int_a^b v_z r^2 \sin\theta\, dr\, d\theta\, d\phi.$$

If A were zero, we would get $p = 0$, as is seen by integrating over θ first. The net contribution is completely determined by A, and indeed

$$p = -\rho\int_0^{2\pi}\int_0^{\pi}\int_a^b A r^2 \sin\theta\, dr d\theta\, d\phi = -{^4/_3}\,\pi\rho A(b^3 - a^3) = -\,{^4/_3}\,\pi\rho a^3\, v_0 \tag{13}$$

by equation (5). Thus the momentum of the fluid is, indeed, equal to and opposite in sign to that of a sphere of the same density. It is of interest that, had we gone to the limit of an infinite fluid and set $A = 0$ before setting up the integral, we would have missed this momentum altogether. The contributions always come predominantly from near the wall of the container.

If a force f acts on the small sphere through a very small distance δz, the work done will appear as kinetic energy of the sphere and the fluid. We will thus have for a sphere of mass m having the same density as the liquid

$$f\delta z = \delta({^3/_2}\cdot{^1/_2}\, mv_0^2) \approx {^3/_2}\, mv_0\, \delta v_0. \tag{14}$$

The time through which the force acts is $\delta z/v_0$. Thus the change of momentum is ${^3/_2}\, m\delta v_0$, and the moving sphere has the same effective mass for momentum as it does for kinetic energy. But we have seen that if the large sphere is held fixed, the momentum of the liquid cancels that of the sphere. The momentum must therefore be transmitted to the container through pressure exerted upon the walls. We intend to calculate this pressure to see how this occurs. In classical treatises on hydrodynamics[4] it has been shown that the pressure is given by

$$P = \rho\left(\frac{\partial\Phi}{\partial t} - {^1/_2}\, v^2\right) + F(t), \tag{15}$$

where $F(t)$ is a constant of integration, which may be a function of time, but which may be ignored for our purposes.

We shall first show that (consistent with our findings about the back momentum of the fluid) if a sphere moves a small distance $v_0\delta t$ from the center of the container in a time δt with uniform velocity, the pressure on the container in the forward direction will be equal to that in the backward direction. From equations (2) and (4), we see that the value of $|v|$ and hence of v^2 will be the same at the angle θ as at the angle $\pi - \theta$. Thus the term $\rho v^2/2$ in equation (15) will contribute pressures which will be equal at θ and $\pi - \theta$, and since the z-components of the forces thus produced will be in opposite directions, these forces will balance. It is clear from the discussion of equation (11), that, even if this cancellation did not occur, the contribution from this term would be negligible in a large container.

To find $\partial\Phi/\partial t$ we evaluate Φ after the sphere has moved a distance $v_0\delta t$. We need to maintain the boundary conditions of the surfaces both of the sphere and the container when the sphere is in its new position. Let s be the distance from the center of the sphere, r remaining the distance from the center of the container. Taking s as the independent variable for the Laplace equation, we observe that the solution

$$\Phi = (As + B/s^2) \cos \theta_s, \tag{16}$$

where θ_s is measured from the center of the moving sphere, would, with A and B given by equations (5) and (6), satisfy the boundary conditions at $s = a$, but not at $r = b$. We note that, since $\partial \cos \theta/\partial r = 0$ (we must hold θ constant in differentiating at the surface of the container),

$$\frac{\partial\Phi}{\partial r} = \left(A - \frac{2B}{s^3}\right)\left(1 + \frac{\partial(s-r)}{\partial r}\right)\cos\theta_s + \left(As + \frac{B}{s^2}\right)\frac{\partial}{\partial r}(\cos\theta_s - \cos\theta), \tag{17}$$

where we have written $\partial s/\partial r = 1 + \partial(s - r)/\partial r$. From the law of cosines for the triangle s, r, $v_0\delta t$ (the angles between r and $v_0\delta t$ and s and $v_0\delta t$ being θ and $\pi - \theta_s$, respectively),

$$s^2 = r^2 + (v_0\delta t)^2 - 2rv_0\delta t \cos \theta,$$

or

$$r^2 = s^2 + (v_0\delta t)^2 + 2sv_0\delta t \cos \theta_s,$$

we find to the first order in $v_0\delta t$

$$s - r = -v_0\delta t \cos \theta, \tag{18}$$

and

$$s \cos \theta_s - r \cos \theta = -v_0\delta t.$$

The latter becomes, using equation (18),

$$\cos \theta_s - \cos \theta = -(v_0\delta t/r)(1 - \cos^2 \theta). \tag{19}$$

From (18) we see that $\partial(s - r)/\partial r \approx 0$. It is also true, when $s \approx b$, that $A - 2B/s^3 \approx 0$. However, it is important to evaluate this more precisely. With the aid of equation (18), we see that when $r = b$,

$$A - \frac{2B}{s^3} = A - \frac{2B}{b^3 + s^3 - b^3} \approx A - \frac{2B}{b^3 - 3b^2v_0\delta t \cos\theta} \approx -6v_0\delta t(B/b^4)\cos\theta.$$

Using this expression and equation (19) in equation (17), and noting again that $A = 2B/b^3$, we find

$$(\partial\Phi/\partial r)_{r=b} \approx 3v_0\delta t(B/b^4)(1 - 3\cos^2\theta).$$

We now seek a solution of equation (1) for which $(\partial\Phi/\partial r)_{r=b} = 0$ (to the first order of $v_0\delta t$), while still conforming to $(\partial\Phi/\partial s)_{s=a} = v_0 \cos\theta_s$. Such a solution can readily be seen to be

$$\Phi = (As + B/s^2)\cos\theta_s + C(s^2 - 2a^5/3s^3)(1 - 3\cos^2\theta_s). \tag{20}$$

It satisfies the condition at $s = a$, because there the derivative with respect to s (θ_s constant) of the last term vanishes. When $s \approx b$, we may, in the last term, replace s by r and $\cos\theta_s$ by $\cos\theta$, since C must itself be small. Then it can be seen that if $b \gg a$, the boundary condition is satisfied if $C \approx -{}^3/_2\, v_0\delta tB/b^5 \approx -{}^3/_4\, v_0{}^2\delta t\, a^3/b^5$. The largest term in $d\Phi/dt$, then, for $b \gg a$, is $\sim v_0{}^2\, a^3/b^3$ at $r = b$, but, in any case, all terms are multiplied by either $\cos^2\theta$ or 1, so the directional balancing of the force on the surface of the container occurs as for the $(\rho/2)v^2$ term of equation (15).

But suppose that the sphere, located at the center of the container, receives an impulse, so that v_0 changes by δv_0 in time δt. In this case we need only to differentiate A and B, equations (5) and (6), and use equation (4), to find, at $r = b$,

$$\partial\Phi/\partial t = {}^3/_2(a^3/b^2)\cos\theta\,(dv_0/dt)\cdot b^3/(b^3 - a^3), \tag{21}$$

which is antisymmetric fore and aft.

The net force on the container is obtained by multiplying $\partial\Phi/\partial t$ by the element of area, $b^2 \sin\theta\, d\theta\, d\phi$, and by $\cos\theta$ to get the z-component, or

$$f_z = {}^3/_2\,\rho a^3(dv_0/dt)\int_0^{2\pi}\int_0^{\pi} \cos{}^2\theta \sin\theta\, d\theta\, d\phi\cdot b^3/(b^3 - a^3)$$
$$= {}^3/_2\cdot{}^4/_3\pi\rho a^3(dv_0/dt)\cdot b^3/(b^3 - a^3). \tag{22}$$

Thus the force is that to produce the given acceleration in a body with a mass equal to ${}^3/_2$ the mass of the sphere, with the same correction for the size of the container as appears in (12), and the corresponding momentum will be transferred to the system as a whole.[5]

If the sphere has a density ρ' different from that of the liquid, it will respond to an impulse as though it had a mass of ${}^4/_3\pi a^3(\rho' + {}^1/_2\rho)$. Thus the momentum transmitted to the container will not be equal to the momentum transmitted to the sphere, for the former will, as before, be ${}^4/_3\pi a^3({}^3/_2\rho)$. The difference will be reflected in a forward momentum of the sphere differing from the back momentum of the fluid by ${}^4/_3\pi a^3(\rho' - \rho)v_0$. Thus some of the total momentum may be considered to be separate momentum of the sphere as contrasted to momentum of the system as a whole; however, the displacement δx through which the impulse acts is related to the time in the same way as before: $\delta x = v_0\delta t$.

However, the last relationship will not be true if the impulse produces a change in *internal* energy of the sphere. Only the part transmitted to kinetic energy will be directly related to momentum and, as before, the change in momentum of the whole system will, regardless of the mass and internal energy of the sphere, be $^3/_2 \cdot {}^4/_3 \pi \rho a^3 \delta v_0 \cdot b^3/(b^3 - a^3)$.

Though some of these observations may be applicable to the motion of an ^{3}He atom dissolved in ^{4}He liquid, or to the motion of a roton in liquid ^{4}He, there will be obvious differences. One difference will arise because liquid helium is not incompressible, so the pressure changes resulting from a change in the motion of the particle under consideration, due, for example, to collision with a neutron, will not be transmitted instantaneously. This may not be too serious since (15) is a good approximation even if the liquid is compressible, provided ρ does not vary too much, and the use of the velocity potential depends only on the motion's being irrotational, although the Laplacian cannot be equated to zero if the liquid is compressible. If, however, the use of Laplace's equation gives a reasonable approximation near the boundary of the container before and after the change of motion has occurred, we may expect the change $\delta\Phi$ in Φ to be a reasonable substitute for $(\partial\Phi/\partial t)\ \delta t$ in calculating the total change in momentum from (15). Furthermore, we may expect the change in Φ produced by the acceleration due to collision with a neutron to have the most important effect. To see this, we compare the value of $d\Phi/dt$ from (21) with the value $v_0^2a^3/b^3$, which would be obtained in the case of uniform motion, if we ignored the cancellation due to directional effects. This involves a comparison of v_0^2/a with dv_0/dt. Applying this to an atom moving in a superfluid, a/v_0 is approximately the time required for the atom to move an atomic distance, and $v_0 \div a/v_0$ is an "acceleration" which is certainly much smaller than the acceleration produced by a neutron colliding with a helium nucleus; furthermore, in $v_0^2a^3/b^3$ it is multiplied by a/b.

On the other hand, the molecular character of the liquid means that the equations cannot be applied close to a single atom and this may have an effect on the effective mass. A roton in liquid helium may be expected to have some of the characteristics of a vibration of one of the atoms in the field of its neighbors. Thus some of the energy will go into this internal energy, and the remarks of the paragraph before the last will apply—only part of the energy will be associated with the momentum. If the energy is high enough for this vibrator to dissociate, i.e., for one of the atoms to break away from its neighbors, the analogy to a moving sphere should be closest.

The effective mass m may be calculated from the relation between momentum and total kinetic energy ϵ_k of sphere and fluid, namely,

$$2\ m = (mv)^2/{}^1/_2mv^2 = p^2/\epsilon_k, \tag{23}$$

but cannot, apparently, be obtained for rotons with high p and ϵ_k. The curve obtained by neutron excitation is the curve exhibited in Figure 1. The part just to the right of the minimum parallels the phonon part of the curve almost exactly. This may mean that an energetic neutron produces a roton and a phonon rather than a more energetic roton. One may, however, apply equation (23) to the minimum of the curve, taking ϵ_k as the total energy of the roton. The data[2] give

$$m - m_4 \approx 1.5\ m_4, \tag{24}$$

where m_4 is the mass of ^{4}He. This is already larger than might be expected for a sphere moving in a continuum, but the value of m must be larger than this, since only part of the energy transmitted shows as kinetic energy. It is of interest that this bears no resemblance to the apparent mass of a roton obtained by dividing the mass of normal fluid arising from rotons by the number of rotons. The latter depends on the temperature and has been described in different ways.[6, 7] It is involved in a different process, the motion of superfluid relative to normal fluid, which is not directly connected with transfer of momentum to the container. A roton is presumably formed by direct collision of a neutron with a helium atom. The latter could then behave momentarily like a sphere moving through the superfluid. If the impulse associated with the kinetic energy is transmitted to the surrounding liquid before the internal vibrational part of the energy of the roton is transmitted, exciton fashion, to neighboring atoms, it seems reasonable to treat the excited molecule as a single body moving in a nearly perfect fluid.[7] This would be expected to be a better description, the higher the momentum.

In the case of ^{3}He in ^{4}He, there is no possibility of transmission of the excitation from atom to atom, the ^{3}He does move as a single body. Therefore, the effective mass can be obtained from second sound measurements[8] in solutions of ^{3}He in ^{4}He, or from the onset of Fermi degeneracy as evidenced by specific heat measurements in such solutions.[9] The former measurements give 2.8 m_3, the latter 2.34 m_3, where m_3 is the mass of ^{3}He. Translated into terms comparable with equation (24), these figures give

$$m - m_3 \approx 1.35\ m_4 \tag{25a}$$

and

$$m - m_3 \approx 1.00\ m_4. \tag{25b}$$

These figures are somewhat smaller than the value shown in equation (24). This means that the ^{3}He atom has less effect on the surrounding medium than does the ^{4}He atom.

The fact that the values in equations (24) and (25) are larger than $^1/_2$ is not surprising even from the most elementary point of view, since each atom actually has an exclusion sphere which has a radius twice the actual radius of the atom. Thus an atom may effectively occupy a larger volume as a moving sphere than its atomic volume as obtained from the density. The larger value which appears in equation (24) may have something to do with the differences between a ^{4}He excitation and a ^{3}He impurity, which have already been mentioned. It occurs in spite of the fact that, on account of its larger zero-point motion, the apparent molecular volume of a ^{3}He atom in liquid ^{4}He is 1.28 times that of the ^{4}He atoms.[10]

A recent theoretical calculation gives[11] $m - m_3 = 0.64\ m_4$.

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